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Third-Order Equation for Bispinors

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Abstract

We present the general features of a bispinor field that obeys a third-order equation. It separates into two massive fields that obey the Dirac equation and a four-component massless field. We discuss briefly its electromagnetic interactions and a leptonic interaction that introduces a mass difference. This field can thus describe the electron, the muon and both neutrinos. The difficulties related to inconsistencies between electromagnetic and weak interactions for the two-component spinors are still present for the bispinor field.

1. *Introduction*

Recent changes in our understanding of leptons and weak interactions make an examination of the less common wave equations worthwhile.

In a previous paper (Marx, 1974), we worked with a third-order equation for two-component spinors. We now do a similar study of the corresponding equation for bispinors, mentioned before by Kibble & Polkinghorne (1958) more or less in passing. This equation has the right number of degrees of freedom to accommodate the electron, the muon and the two neutrinos. The basic equation has a single mass parameter, but it is a simple matter to change it to include two different masses.

When the interaction with the electromagnetic field is introduced via the usual gauge-invariant substitution, the massless field appears to be charged. Nevertheless, since its 'charge' is separately conserved, this is not an insurmountable objection. As far as weak interactions are concerned, the number of possible interaction Lagrangian densities that can be constructed from the field and its first and second derivatives is quite large. We limit ourselves to developing in some detail a two-fermion pseudoscalar term that leads to the splitting of the masses, which can be regarded either as an interaction or as a change in the free-field equation.

We study the free-field equation in Section 2, and introduce electromagnetic and leptonic interactions in Sections 3 and 4 respectively, plus some concluding remarks in Section 5.

We use natural units and a time-favoring metric in a real Minkowski space; the notation used here follows closely that in Marx (1974) and some earlier papers mentioned there.

2. *The Free Field*

We use as a starting point a Lagrangian density analogous to the one in Marx (1974),

$$\mathscr{L}_0 = \tfrac{1}{2}i[(\bar{\psi}_{,\alpha}\gamma_\alpha\gamma_\mu\gamma_\beta\psi_{,\beta\mu} - \bar{\psi}_{,\alpha\mu}\gamma_\alpha\gamma_\mu\gamma_\beta\psi_{,\beta})/m^2 - \bar{\psi}\gamma_\alpha\psi_{,\alpha} + \bar{\psi}_{,\alpha}\gamma_\alpha\psi] \tag{2.1}$$

which yields the third-order equation of motion

$$-i\gamma\cdot\partial(\partial^2 + m^2)\psi = 0 \tag{2.2}$$

and the conserved current and energy-momentum densities

$$j'_\mu = \tfrac{1}{2}[(\bar{\psi}_{,\mu}\gamma_\alpha\psi_{,\alpha} - \bar{\psi}\gamma_\mu\psi_{,\alpha\alpha} - \bar{\psi}\gamma_\alpha\psi_{,\alpha\mu})/m^2 - \bar{\psi}\gamma_\mu\psi] + \text{c.c.} \tag{2.3}$$

$$T'_{\mu\nu} = \tfrac{1}{2}i[\bar{\psi}_{,\nu}(\gamma_\mu\psi_{,\alpha\alpha} + m^2\gamma_\mu\psi + \gamma_\alpha\psi_{,\alpha\mu})/m^2 - \bar{\psi}_{,\mu\nu}\gamma_\alpha\psi_{,\alpha}/m^2] + \text{c.c.} - \mathscr{L}_0 g_{\mu\nu} \tag{2.4}$$

where c.c. stands for the complex conjugate of the preceding term.

Following procedures used before (Marx, 1967, 1974), we decompose the field ψ into three parts

$$\psi = \eta + \xi + \zeta \tag{2.5}$$

defined by the projections in the space of solutions of equation (2.2),

$$\eta = (1 + \partial^2/m^2)\psi \tag{2.6}$$

$$\xi = (1/2m^2)(im\gamma\cdot\partial - \partial^2)\psi \tag{2.7}$$

$$\zeta = (1/2m^2)(-im\gamma\cdot\partial - \partial^2)\psi \tag{2.8}$$

These fields then satisfy

$$-i\gamma\cdot\partial\eta = 0 \tag{2.9}$$

$$(-i\gamma\cdot\partial + m)\xi = 0 \tag{2.10}$$

$$(-i\gamma\cdot\partial - m)\zeta = 0 \tag{2.11}$$

To separate the contributions from the different fields, we add a term $f_{\alpha\mu,\alpha}$ to j'_μ where†

$$f_{\alpha\mu} = -(i/2m^2)\bar{\psi}\sigma_{\alpha\mu}\gamma_\beta\psi_{,\beta} + \text{c.c.} \tag{2.12}$$

$$\sigma_{\alpha\mu} = \tfrac{1}{2}i(\gamma_\alpha\gamma_\mu - \gamma_\mu\gamma_\alpha) \tag{2.13}$$

We obtain

$$j_\mu = 2(\bar{\xi}\gamma_\mu\xi + \bar{\zeta}\gamma_\mu\zeta) - \bar{\eta}\gamma_\mu\eta \tag{2.14}$$

and we note that the contribution from the fields ξ and ζ have the same sign, which was not the case in Marx (1967).

Similarly, we add a term $f_{\alpha\mu\nu,\alpha}$ to $T'_{\mu\nu}$, where

$$f_{\alpha\mu\nu} = -(1/2m^2)\bar{\psi}_{,\nu}\sigma_{\alpha\mu}\gamma_\beta\psi_{,\beta} - \tfrac{1}{2}i\bar{\eta}(g_{\mu\nu}\gamma_\alpha - g_{\alpha\nu}\gamma_\mu)(\xi + \zeta) + \text{c.c.} \tag{2.15}$$

and obtain

$$T_{\mu\nu} = \tfrac{1}{2}i[2(\bar{\xi}\gamma_\mu\xi_{,\nu} - \bar{\xi}_{,\nu}\gamma_\mu\xi + \bar{\zeta}\gamma_\mu\zeta_{,\nu} - \bar{\zeta}_{,\nu}\gamma_\mu\zeta) - (\bar{\eta}\gamma_\mu\eta_{,\nu} - \bar{\eta}_{,\nu}\gamma_\mu\eta)] \tag{2.16}$$

The solutions to equations (2.9) through (2.11) can be expanded in terms of momentum-space amplitudes,

$$\eta = (2\pi)^{-3/2}\int d^3k u_{0\lambda}(\mathbf{k})[a_\lambda(\mathbf{k})\exp(-ik\cdot x) + c_\lambda(\mathbf{k})\exp(ik\cdot x)] \tag{2.17}$$

$$\xi = (2\pi)^{-3/2}\int d^3p(m/2E)^{1/2}[u_\lambda(\mathbf{p})b_\lambda(\mathbf{p})\exp(-ip\cdot x) + v_\lambda(\mathbf{p})d_\lambda(\mathbf{p})\exp(ip\cdot x)] \tag{2.18}$$

$$\zeta = (2\pi)^{-3/2}\int d^3p(m/2E)^{1/2}[v_\lambda(\mathbf{p})f_\lambda(\mathbf{p})\exp(-ip\cdot x) + v_\lambda(\mathbf{p})g_\lambda(\mathbf{p})\exp(ip\cdot x)] \tag{2.19}$$

where u_λ and v_λ, collectively designated by w_λ, are given by

$$w_\lambda(\mathbf{p}) = [(E+m)/2m]^{1/2}[1 + \boldsymbol{\alpha}\cdot\mathbf{p}/(E+m)]\, w_\lambda^{(0)}(\hat{p}) \tag{2.20}$$

$$u_\lambda^{(0)}(\hat{p}) = \begin{pmatrix}\chi_\lambda(\hat{p})\\ 0\end{pmatrix}, \qquad v_\lambda^{(0)}(\hat{p}) = \begin{pmatrix}0\\ \chi_{-\lambda}(\hat{p})\end{pmatrix} \tag{2.21}$$

the χ_λ being the two-component helicity states and

$$u_{0\lambda}(\mathbf{k}) = 2^{-1/2}(1 + \boldsymbol{\alpha}\cdot\hat{k})u_\lambda^{(0)}(\hat{k}) \tag{2.22}$$

$$E = p_0 = (\mathbf{p}^2 + m^2)^{1/2} \tag{2.23}$$

$$k_0 = |\mathbf{k}| \tag{2.24}$$

† The definition of $\sigma_{\alpha\mu}$ in Marx (1967) differs from the present one by an overall change of sign.

The total 'charge' can then be expressed as

$$Q = \int d^3p\,[b_\lambda^* b_\lambda + d_\lambda^* d_\lambda + f_\lambda^* f_\lambda + g_\lambda^* g_\lambda] - \int d^3k\,[a_\lambda^* a_\lambda + c_\lambda^* c_\lambda] \tag{2.25}$$

and the energy-momentum vector is

$$P_\mu = \int d^3p\,p_\mu\,[b_\lambda^* b_\lambda - d_\lambda^* d_\lambda + f_\lambda^* f_\lambda - g_\lambda^* g_\lambda] - \int d^3k\,k_\mu\,[a_\lambda^* a_\lambda - c_\lambda^* c_\lambda] \tag{2.26}$$

The fields ξ and ζ represent spin-$\frac{1}{2}$ particles of mass m which play effectively the same role in spite of the different signs of the mass in equations (2.10) and (2.11). The field η represents a massless spin-$\frac{1}{2}$ particle, and both helicities are present, whence it can describe the two types of neutrinos and the corresponding antineutrinos.

In a quantization we have to decide whether d_λ and g_λ are annihilation operators, as in Marx (1972), or whether they are creation operators, which is the usual choice. In the latter case, normal ordering changes the signs of the corresponding terms in the expressions (2.25) and (2.26) for Q and P_μ.

A different separation of the field ψ into component fields can be performed by going back to two-component spinors. The upper and lower components of ψ in the third-order equation (2.2) are uncoupled, and they obey the equations presented in Marx (1974),

$$i\partial^{\dot{B}A}(\partial^2 + m^2)\varphi_A = 0 \tag{2.27}$$

$$i\partial_{\dot{A}B}(\partial^2 + m^2)\chi^{\dot{A}} = 0 \tag{2.28}$$

In any case, the number of degrees of freedom present in the field ψ is the right one to describe all leptons.

3. *Electromagnetic Interactions*

These interactions can be introduced by the usual gauge-invariant substitution

$$\partial_\mu \to D_\mu = \partial_\mu - ieA_\mu \tag{3.1}$$

so that the Lagrangian density becomes

$$\mathscr{L} = \tfrac{1}{2}i\{[(D_\alpha^*\bar{\psi})\gamma_\alpha\gamma_\mu\gamma_\beta D_\mu D_\beta\psi - (D_\mu^* D_\alpha^*\bar{\psi})\gamma_\alpha\gamma_\mu\gamma_\beta D_\beta\psi]/m^2 - \bar{\psi}\gamma_\alpha D_\alpha\psi + (D_\alpha^*\bar{\psi})\gamma_\alpha\psi\} \tag{3.2}$$

The resulting equation of motion is

$$i\gamma\cdot D[(\gamma\cdot D)^2 + m^2]\,\psi = 0 \tag{3.3}$$

and the solutions can still be decomposed as in equation (2.5) into parts that satisfy

$$-i\gamma \cdot D\eta = 0 \tag{3.4}$$

$$(-i\gamma \cdot D + m)\xi = 0 \tag{3.5}$$

$$(-i\gamma \cdot D - m)\zeta = 0 \tag{3.6}$$

As was the case in Marx (1974), the massless part of the field appears to be charged, a consequence of the gauge covariance of ψ. The precise behavior of a massless charged field remains to be investigated. The three terms in the current density (2.14) are separately conserved even in the presence of electromagnetic interactions, because the fields obey the uncoupled equations (3.4) to (3.6).

We have some further remarks on the electromagnetic interactions at the end of the following section.

4. *Lepton Interactions*

If we want to construct an interaction Lagrangian from the field and its first and second derivatives we are faced with a choice among a large number of terms that can be used singly or in combination.

An obvious candidate for weak interactions is the four-fermion point coupling,

$$\mathscr{L}_I' = G\bar{\psi}\gamma_\mu(1 + i\gamma_5)\psi\bar{\psi}\gamma_\mu(1 + i\gamma_5)\psi \tag{4.1}$$

which contains the usual terms plus some additional ones.

On the other hand, to construct a theory that describes actual leptons we have to introduce the mass difference between the electron and the muon. It is possible to treat this through an added interaction Lagrangian density, which can also be made a part of $\mathscr{L}_0$, of course.

For this purpose, we choose a pseudoscalar quantity, as suggested by the parity non-conservation of weak interactions.

$$\mathscr{L}_I = \tfrac{1}{2}g(\bar{\psi}\gamma_5\gamma_\mu\psi_{,\mu} + \bar{\psi}_{,\mu}\gamma_\mu\gamma_5\psi) \tag{4.2}$$

The new equation of motion is

$$-i\gamma \cdot \partial(\partial^2 + m^2)\psi = -gm^2\gamma_5\gamma \cdot \partial\psi \tag{4.3}$$

which can be rewritten as

$$-i\gamma \cdot \partial[\partial^2 + m^2(1 - ig\gamma_5)]\,\psi = 0 \tag{4.4}$$

In the representation of the γ_μ that arises naturally when the starting point is two-component spinors (Marx, 1974), we have

$$\gamma_5 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \tag{4.5}$$

where each element represents a 2 x 2 matrix, a multiple of either the zero or unit matrix. Thus, the upper and lower components of ψ obey equations (2.27) and (2.28) with masses

$$m' = m(1+g)^{1/2} \tag{4.6}$$

$$m'' = m(1-g)^{1/2} \tag{4.7}$$

respectively.

A different way of approaching this 'interaction' comes from the decomposition of the field ψ indicated in equations (2.5) through (2.8). The fields now obey

$$(-i\gamma\cdot\partial + m)\xi = \tfrac{1}{2}g\gamma_5\gamma\cdot\partial\psi \tag{4.8}$$

$$(-i\gamma\cdot\partial - m)\zeta = \tfrac{1}{2}g\gamma_5\gamma\cdot\partial\psi \tag{4.9}$$

$$-i\gamma\cdot\partial\eta = -g\gamma_5\gamma\cdot\partial\psi \tag{4.10}$$

which can be rewritten as

$$-i\gamma\cdot\partial A\Psi + mB\Psi = 0 \tag{4.11}$$

where

$$A = \begin{pmatrix} 1+\frac{1}{2}ig\gamma_5 & \frac{1}{2}ig\gamma_5 & \frac{1}{2}ig\gamma_5 \\ \frac{1}{2}ig\gamma_5 & 1+\frac{1}{2}ig\gamma_5 & \frac{1}{2}ig\gamma_5 \\ -ig\gamma_5 & -ig\gamma_5 & 1-ig\gamma_5 \end{pmatrix} \tag{4.12}$$

$$B = \operatorname{diag}(I, -I, 0) \tag{4.13}$$

$$\tilde{\Psi} = (\xi, \zeta, \eta) \tag{4.14}$$

The new set of fields

$$\Psi' = A\Psi \tag{4.15}$$

obeys

$$(-i\Gamma\cdot\partial + mB')\Psi' = 0 \tag{4.16}$$

where

$$\Gamma = \operatorname{diag}(\gamma, \gamma, \gamma) \tag{4.17}$$

$$B' = \begin{pmatrix} 1-\frac{1}{2}ig\gamma_5 & -\frac{1}{2}ig\gamma_5 & -\frac{1}{2}ig\gamma_5 \\ \frac{1}{2}ig\gamma_5 & -1+\frac{1}{2}ig\gamma_5 & \frac{1}{2}ig\gamma_5 \\ 0 & 0 & 0 \end{pmatrix} \tag{4.18}$$

The solutions of the set of linear equations (4.16) can be expressed in terms of plane waves

$$\Psi'(x) = \Phi(p)\exp(-ip\cdot x) \tag{4.19}$$

whence Φ satisfies

$$(-\Gamma \cdot p + mB')\Phi = 0 \tag{4.20}$$

which has non-trivial solutions when the secular equation

$$\det(-\Gamma \cdot p + mB') = 0 \tag{4.21}$$

is satisfied. This equation reduces to

$$\{p^2 [p^2 - m^2(1+g)] [p^2 - m^2(1-g)]\}^2 = 0 \tag{4.22}$$

confirming the expected mass spectrum with masses 0, m' and m''.

The field η' is a free massless field, while ξ' and ζ' are coupled to each other and to η'. When η' vanishes the linear combinations of ξ' and ζ' that correspond to masses m' and m'' are proportional to

$$\Phi'_+(\mathbf{p}) = \begin{pmatrix} (1 + ir'\gamma_5)u'_\lambda(\mathbf{p}) \\ (-r' - i\gamma_5)u'_\lambda(\mathbf{p}) \end{pmatrix} \tag{4.23}$$

$$\Phi'_-(\mathbf{p}) = \begin{pmatrix} (-r' - i\gamma_5)v'_\lambda(\mathbf{p}) \\ (1 + ir'\gamma_5)v'_\lambda(\mathbf{p}) \end{pmatrix} \tag{4.24}$$

$$\Phi''_+(\mathbf{p}) = \begin{pmatrix} (1 + ir''\gamma_5)u''_\lambda(\mathbf{p}) \\ (r'' + i\gamma_5)u''_\lambda(\mathbf{p}) \end{pmatrix} \tag{4.25}$$

$$\Phi''_-(\mathbf{p}) = \begin{pmatrix} (r'' + i\gamma_5)v''_\lambda(\mathbf{p}) \\ (1 + ir''\gamma_5)v''_\lambda(\mathbf{p}) \end{pmatrix} \tag{4.26}$$

where

$$r' = \frac{m' - m}{m' + m}, \qquad r'' = \frac{m - m''}{m + m''} \tag{4.27}$$

and u'_λ, v'_λ, u''_λ and v''_λ are given by equation (2.20) with masses m' and m''.

If we introduce an electromagnetic interaction through a modification of equation (4.16),

$$(-i\Gamma \cdot D + mB')\Psi'(x) = 0 \tag{4.28}$$

the massless field η' still obeys the uncoupled equation (3.4). Or we can start again from the Lagrangian density (3.2) adding the corresponding terms from that in equation (4.2), that is,

$$\mathscr{L}''_I = \tfrac{1}{2}g[\bar{\psi}\gamma_5\gamma_\mu D_\mu\psi + (D^*_\mu\bar{\psi})\gamma_\mu\gamma_5\psi] \tag{4.29}$$

leading to

$$-i\gamma \cdot D[(\gamma \cdot D)^2 + m^2(1 - ig\gamma_5)]\,\psi = 0 \tag{4.30}$$

and we can proceed from here to equation (4.28).

5. *Concluding Remarks*

This bispinor field we have presented has the attractive feature that it describes simultaneously two massive fermion fields that obey the Dirac equation and a massless one, thus having the right number of degrees of freedom for all leptons. We have also shown how the mass difference between the electron and the muon can be introduced through a pseudoscalar interaction term. The electromagnetic interactions present the difficulty of having the massless field appear charged, and it is not readily compatible with more complicated lepton interactions.

This field can be used either in the context of relativistic quantum mechanics or of quantum theory of fields. In the latter case, the propagator will give much better convergence of the terms in a perturbation expansion than the Dirac field.

We have not explored the many other possible interaction terms both leptonic and with the electromagnetic field. This can be done in case there are further indications that such a common lepton field is desirable.

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